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A NOTE ON DOUBLE CENTRAL EXTENSIONS IN EXACT MAL'TSEV CATEGORIES

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Dedicated to Francis Borceux on the occasion of his sixtieth birthday

Abstract: The characterisation of double central extensions in terms of commutators due to Janelidze (in the case of groups), Gran and Rossi (in the case of Mal'tsev varieties) and Rodelo and Van der Linden (in the case of semi-abelian categories) is shown to be still valid in the context of exact Mal'tsev categories.

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In his article [7], George Janelidze gave a characterisation of the double central extensions of groups in terms of commutators. Not only did he thus relate Galois theory to commutator theory, but he also sowed the seeds for a new approach to homological algebra, where higher-dimensional (central) extensions are used as a basic tool—see, for instance, [3, 4, 8, 11].

Expressed in terms of commutators of equivalence relations [10, 12], his result amounts to the following: a double extension

$$\begin{tikzcd} X \arrow[r, "c"] \arrow[d, "d"'] & C \arrow[d] \\ D \arrow[r] & Z \end{tikzcd} \qquad (\mathbf{A})$$

is central if and only if $[R[d], R[c]] = \Delta_X = [R[d] \cap R[c], \nabla_X]$. Here $R[d]$ and $R[c]$ denote the kernel pairs of d and c, and Δ_X and ∇_X are the smallest and the largest equivalence relation on X. This characterisation was generalised twice, first to the context of Mal'tsev varieties (by Marino Gran and Valentina Rossi [6]) and then to semi-abelian categories (by Diana Rodelo and Tim Van der Linden [11]), but both generalisations are in some way unsatisfactory. Although one of the implications (the "only if"-part) of the proof given in [6] is entirely categorical and easily seen to be valid in any exact Mal'tsev category, the other implication is not, and makes heavy use of

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universal-algebraic machinery. And though the argument worked out in [11] is conceptual and does not depend on a varietal setting, the context is narrowed to that of semi-abelian categories—which is too small to contain all Mal'tsev varieties.

The aim of this note is to improve the situation by providing a proof in what seems to be the most natural context—that of finitely cocomplete Barr exact Mal'tsev categories [2]—and thus unifying the two existing generalisations. In order to avoid endless repetition we chose not to present it in a self-contained manner, but to rely on the results and definitions from [6]. However, in our proof we shall have to consider also three-fold and four-fold extensions, so that a few introductory words on higher-dimensional extensions cannot be avoided. For an in-depth discussion on this subject we refer the reader to [4] and [3].

Consider a finitely cocomplete Barr exact Mal'tsev category $\mathcal{A}$. Given $n \geq 0$, denote by $\mathsf{Arr}^n\mathcal{A}$ the category of n-dimensional arrows in $\mathcal{A}$. (A zero-dimensional arrow is an object of $\mathcal{A}$.) n-fold extensions are defined inductively as follows. A **(one-fold) extension** is a regular epimorphism in $\mathcal{A}$. For $n \geq 1$, an $(n+1)$**-fold extension** is a commutative square $\mathbf{A}$ in $\mathsf{Arr}^{n-1}\mathcal{A}$ (an arrow in $\mathsf{Arr}^n\mathcal{A}$) such that in the induced commutative diagram

$$\begin{tikzcd}
X \arrow[dr] \arrow[drr, bend left, "c"] \arrow[ddr, bend right, "d"'] & & \\
& D \times_Z C \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d] \\
& D \arrow[r] & Z
\end{tikzcd}$$

every arrow is an n-fold extension. Thus for $n = 2$ we regain the notion of double extension.

Recall that a commutative square of extensions is a double extension if and only if its kernel pair in $\mathsf{Arr}\mathcal{A}$ is an extension; see, for instance, [1]. Since the concept of double extension is symmetric, this has the following consequences:

- double extensions are stable under composition;
- if a composite $g \circ f\colon A \to B \to C$ of arrows in $\mathsf{Arr}\mathcal{A}$ is a double extension and B is an extension, then $g\colon B \to C$ is a double extension;
- any split epimorphism of extensions is a double extension.

And then also the following is straightforward to prove:

- the pullback in $\mathsf{Arr}\mathcal{A}$ of a double extension $A \to B$ along a double extension $C \to B$ is a double extension.

In fact, for *any* $n \geq 1$, a commutative square in $\mathsf{Arr}^{n-1}\mathcal{A}$ consisting of n-fold extensions is an $(n+1)$-fold extension if and only if its kernel pair is an n-fold extension, and thus for all of the above listed properties one obtains higher dimensional versions as well. This is easily shown by induction.

Three-fold extensions are of great value in understanding the behaviour of the reflector $\mathsf{centr}\colon \mathsf{Ext}\mathcal{A} \to \mathsf{CExt}\mathcal{A}$ which sends an extension $f\colon A \to B$ to its centralisation $\mathsf{centr}f = A/[R[f], \nabla_A] \to B$. (Here we denoted by $\mathsf{Ext}\mathcal{A}$ and $\mathsf{CExt}\mathcal{A}$ the full subcategories of $\mathsf{Arr}\mathcal{A}$ determined by all extensions and all central extensions, respectively. We shall furthermore write η^1 for the unit of the adjunction.) One has the following property, which is a consequence of the fact that the commutator of equivalence relations is preserved by regular images: for any double extension $f\colon A \to B$, the induced square in $\mathsf{Arr}\mathcal{A}$

$$\begin{tikzcd} A \arrow[r, "\eta^1_A"] \arrow[d, "f"'] & \mathsf{centr}A \arrow[d] \\ B \arrow[r, "\eta^1_B"'] & \mathsf{centr}B \end{tikzcd} \qquad (\mathbf{B})$$

is a three-fold extension. Using the terminology of [3, 4] this means that $\mathsf{CExt}\mathcal{A}$ is a strongly $\mathcal{E}^1$-Birkhoff subcategory of $\mathsf{Ext}\mathcal{A}$, where $\mathcal{E}^1$ denotes the class of all double extensions. To indicate how strong this property is, let us mention here the following consequences, all of which are easily proved, and which are well-known in the case of one-fold extensions [9]. Recall that a double extension $f\colon A \to B$ is **trivial** when the induced square $\mathbf{B}$ is a pullback; it is **normal** when the projections of its kernel pair $R[f]$ are trivial.

- The pullback in $\mathsf{Arr}\mathcal{A}$ of a trivial double extension along a double extension is a trivial double extension;
- the pullback in $\mathsf{Arr}\mathcal{A}$ of a double central extension along a double extension is a double central extension;
- a double central extension that is a split epimorphism in $\mathsf{Arr}\mathcal{A}$ is necessarily trivial.

And it follows that

- the concepts of central and normal double extension coincide.

This last property allows us to prove the next lemma.

Lemma. *A quotient of a double central extension by a three-fold extension is again a double central extension.*

Proof: Consider a three-fold extension, pictured as a square in $\mathsf{Arr}\mathcal{A}$,

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ C \arrow[r, "g"'] & D, \end{tikzcd}$$

and assume that f is a double central extension. Consider the induced diagram of kernel pairs and its reflection into $\mathsf{CExt}\mathcal{A}$.

$$\begin{tikzcd} & \mathsf{centr}R[f] \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dd, dotted] & & \mathsf{centr}A \arrow[dd] \\ R[f] \arrow[ur, "\eta^1_{R[f]}"] \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dd] & & A \arrow[ur, "\eta^1_A"'] \arrow[dd] & \\ & \mathsf{centr}R[g] \arrow[rr, dotted, shift left] \arrow[rr, dotted, shift right] & & \mathsf{centr}C \\ R[g] \arrow[ur, dotted, "\eta^1_{R[g]}"'] \arrow[rr, shift left] \arrow[rr, shift right] & & C \arrow[ur, "\eta^1_C"'] & \end{tikzcd}$$

We focus on the cube of first projections; it is a four-fold extension, since it is a split epimorphism of three-fold extensions. Taking pullbacks in the top and bottom squares of the cube, we obtain the comparison square

$$\begin{tikzcd} R[f] \arrow[r] \arrow[d, "(\eta^1_{R[f]}, p_1)"'] & R[g] \arrow[d, "(\eta^1_{R[g]}, p_1)"] \\ \mathsf{centr}R[f] \times_{\mathsf{centr}A} A \arrow[r] & \mathsf{centr}R[g] \times_{\mathsf{centr}C} C. \end{tikzcd} \tag{C}$$

We just explained why this square $\mathbf{C}$ is a three-fold extension. In particular, it is a pushout in $\mathsf{Arr}\mathcal{A}$. Since f is a normal double extension, the arrow $(\eta^1_{R[f]}, p_1)$ is an isomorphism, hence so is $(\eta^1_{R[g]}, p_1)$. This tells us that g is a normal double extension as well. ■

We are now in a position to prove the characterisation of double central extensions. As mentioned before, we only need to consider one implication: for the other, we refer the reader to [6].

Let $\mathbf{A}$ be a double extension such that $[R[d],R[c]]=\Delta_X=[R[d]\cap R[c],\nabla_X]$. The first condition $[R[d],R[c]]=\Delta_X$ says that there exists a partial Mal'tsev operation $p\colon R[c]\times_X R[d]\to X$. We use the notation $R[d]\square R[c]$ for the largest double equivalence relation on $R[d]$ and $R[c]$, which "consists" of all quadruples $(\alpha,\beta,\delta,\gamma)$ of "elements" of X that satisfy $c(\alpha)=c(\beta),c(\delta)=c(\gamma),d(\alpha)=d(\delta)$ and $d(\beta)=d(\gamma)$. Such a quadruple may be pictured as follows:

$$\begin{bmatrix}\alpha & c & \beta\\ d & & d\\ \delta & c & \gamma\end{bmatrix} \tag{$\mathbf{D}$}$$

Writing

$$\pi\colon R[d]\square R[c]\to R[c]\times_X R[d]$$

for the canonical comparison map (π sends a quadruple $\mathbf{D}$ in $R[d]\square R[c]$ to the triple (α,β,γ)) and $q\colon R[d]\square R[c]\to R[d]\cap R[c]$ for the map which sends a quadruple $\mathbf{D}$ to the couple $(p(\alpha,\beta,\gamma),\delta)$ in $R[d]\cap R[c]$, we obtain the pullback of split epimorphisms

$$\begin{tikzcd}
R[d]\square R[c] \arrow[r, "\pi"] \arrow[d, "q"'] & R[c]\times_X R[d] \arrow[d, "p"] \\
R[d]\cap R[c] \arrow[r, "p_1"'] & X.
\end{tikzcd}$$

Applying the abelianisation functor gives us the next commutative cube, in which the slanted arrows are components of the unit η.

$$\begin{tikzcd}
& \mathsf{ab}(R[d]\square R[c]) \arrow[rr] \arrow[dd, dotted] & & \mathsf{ab}(R[c]\times_X R[d]) \arrow[dd] \\
R[d]\square R[c] \arrow[ur] \arrow[rr] \arrow[dd] & & R[c]\times_X R[d] \arrow[ur] \arrow[dd] & \\
& \mathsf{ab}(R[d]\cap R[c]) \arrow[rr, dotted] & & \mathsf{ab}X \\
R[d]\cap R[c] \arrow[ur, dotted] \arrow[rr] & & X \arrow[ur] &
\end{tikzcd}$$

Since the reflector ab preserves pullbacks of regular epimorphisms along split epimorphisms [5], the back square of this cube is a pullback.

The second condition $[R[d] \cap R[c], \nabla_X] = \Delta_X$ tells us that the extension $(d, c)\colon X \to D \times_X C$ is central. This is equivalent to the projection $p_1\colon R[d] \cap R[c] \to X$ being a trivial extension, which is another way to say that the bottom square in the above cube is a pullback. Hence the two conditions together imply that so is its top square

$$\begin{tikzcd}
R[d]\Box R[c] \arrow[r, "\pi"] \arrow[d, "\eta_{R[d]\Box R[c]}"'] & R[c] \times_X R[d] \arrow[d, "\eta_{R[c]\times_X R[d]}"] \\
\mathsf{ab}(R[d]\Box R[c]) \arrow[r, "\mathsf{ab}\pi"'] & \mathsf{ab}(R[c] \times_X R[d]).
\end{tikzcd}$$

Now consider the left hand side cube and the induced right hand side cube of pullbacks.

$$\begin{tikzcd}
& \mathsf{ab}(R[d]\Box R[c]) \arrow[rr] \arrow[dd, dotted] & & \mathsf{ab}R[d] \arrow[dd] \\
R[d]\Box R[c] \arrow[ru, "\eta_{R[d]\Box R[c]}"] \arrow[rr, "p_2"] \arrow[dd, "p_1"'] & & R[d] \arrow[ru, "\eta_{R[d]}"'] \arrow[dd, "p_1"] & \\
& \mathsf{ab}R[c] \arrow[rr, dotted] & & \mathsf{ab}X \\
R[c] \arrow[ru, dotted, "\eta_{R[c]}"'] \arrow[rr, "p_2"'] & & X \arrow[ru, "\eta_X"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathsf{ab}(R[d]\Box R[c]) \arrow[rr] \arrow[dd, dotted] & & \mathsf{ab}R[d] \arrow[dd] \\
P \arrow[ru] \arrow[rr] \arrow[dd, "\overline{p_1}"'] & & Q \arrow[ru] \arrow[dd, "\overline{p_1}"] & \\
& \mathsf{ab}R[c] \arrow[rr, dotted] & & \mathsf{ab}X \\
R[c] \arrow[ru, dotted, "\eta_{R[c]}"'] \arrow[rr] & & X \arrow[ru, "\eta_X"'] &
\end{tikzcd}$$

Taking into account that, since $R[c] \times_X R[d]$ is a pullback of a split epimorphism along a split epimorphism, $\mathsf{ab}(R[c] \times_X R[d]) = \mathsf{ab}R[c] \times_{\mathsf{ab}X} \mathsf{ab}R[d]$, the foregoing results imply that the left hand side cube is a limit diagram. Hence the comparison square

$$\begin{tikzcd}
R[d]\Box R[c] \arrow[r] \arrow[d] & R[d] \arrow[d] \\
P \arrow[r] & Q
\end{tikzcd}$$

between the two cubes is a pullback, which means that the front square (considered as a horizontal arrow) of the left hand side cube is a trivial double extension. (The vertical arrows p_1 in this double extension are split epimorphims, so their centralisation is their trivialisation—the two arrows $\overline{p_1}$ on the right hand side.) A fortiori, it is a double central extension. Now consider the commutative cube below. Considered as a horizontal arrow, it is a split epimorphism between pullbacks of regular epimorphisms; consequently it is a three-fold extension.

$$\begin{tikzcd}
& R[d]\square R[c] \arrow[rr, "p_2"] \arrow[dl, "p_1"'] \arrow[dd, dotted, "p_2"] & & R[d] \arrow[dl, "p_1"] \arrow[dd, "p_2"] \\
R[c] \arrow[rr, "p_2"] \arrow[ddd] & & X \arrow[ddd, "d"] & \\
& R[c] \arrow[rr, dotted, "p_2"'] \arrow[ddl] & & X \arrow[ddl, "d"] \\
& & & \\
R[f] \arrow[rr, "p_2"'] & & D &
\end{tikzcd}$$

We have just seen that this cube's top square, considered as a horizontal arrow, is a double central extension. Applying the lemma, we find that the bottom square, also considered as a horizontal arrow, is a double central extension as well. But this bottom square is one of the projections of the kernel pair of the double extension $\mathbf{A}$, so that also $\mathbf{A}$ is central, and we obtain:

Theorem. *In a Barr exact Mal'tsev category with finite colimits, a double extension*

$$\begin{tikzcd}
X \arrow[r, "c"] \arrow[d, "d"'] & C \arrow[d] \\
D \arrow[r] & Z
\end{tikzcd}$$

is central if and only if $[R[d], R[c]] = \Delta_X = [R[d] \cap R[c], \nabla_X]$. ■

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